

INTERIOR TRANSMISSION EIGENVALUE PROBLEMS ON COMPACT MANIFOLDS WITH SMOOTH BOUNDARY

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ABSTRACT. In this paper, we consider an interior transmission eigenvalue (ITE) problem on some compact C^∞ -Riemannian manifolds with smooth boundary. In particular, we do not assume that two domains are diffeomorphic, but we impose some conditions of Riemannian metrics and indices of refraction on the boundary. Then we prove the discreteness of the set of ITEs, the existence of infinitely many ITEs, and its Weyl type lower bound. Our problem is non-coercive. However, the argument by Lakshtanov and Vainberg [11] which reduces the problem to the elliptic one using the Dirichlet-to-Neumann map, is valid for manifolds.

1. INTRODUCTION

1.1. Settings of ITE problems on manifolds. We consider two connected and compact C^∞ -Riemannian manifolds (M_1, g_1) and (M_2, g_2) with C^∞ -boundaries ∂M_1 and ∂M_2 , respectively. We assume $d := \dim M_1 = \dim M_2 \geq 2$ and $\dim \Gamma = d - 1$. Throughout of the present paper, we assume that

(A-1) M_1 and M_2 have a common boundary $\Gamma := \partial M_1 = \partial M_2$. Γ is a disjoint union of a finite number of connected and closed components. The metrics satisfy $g_1 = g_2$ on Γ .

We will add some other assumptions for g_1 and g_2 in a neighborhood of the boundary later. Note that we need our geometric assumptions only in some small neighborhoods of the boundary, in particular, we do not assume that M_1 and M_2 are diffeomorphic outside of a small neighborhood of the boundary.

Let Δ_{g_k} , $k = 1, 2$, be the (negative) Laplace-Beltrami operator on each M_k . We consider the following interior transmission eigenvalue (ITE) problem :

$$(1.1) \quad (-\Delta_{g_1} - \lambda n_1)u_1 = 0 \quad \text{in } M_1,$$

$$(1.2) \quad (-\Delta_{g_2} - \lambda n_2)u_2 = 0 \quad \text{in } M_2,$$

$$(1.3) \quad u_1 = u_2, \quad \partial_{\nu_1} u_1 = \partial_{\nu_2} u_2 \quad \text{on } \Gamma,$$

where each $n_k \in C^\infty(\overline{M_k})$, $k = 1, 2$, is strictly positive on M_k . We call $\sqrt{n_k}$ the index of refraction on M_k . If there exists a pair of non-trivial solutions $(u_1, u_2) \in H^2(M_1) \times H^2(M_2)$ of (1.1)-(1.3), we call corresponding $\lambda \in \mathbf{C}$ an *interior transmission eigenvalue*.

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1.2. Backgrounds. ITE problems naturally appears in inverse scattering problems for acoustic wave equations on \mathbf{R}^d with compactly supported inhomogeneity. In \mathbf{R}^d for $d \geq 2$, time harmonic acoustic waves satisfy the equation

$$(1.4) \quad (-\Delta - \lambda n)u = 0, \quad \lambda > 0,$$

where $n \in L^\infty(\mathbf{R}^d)$ is strictly positive in a bounded domain Ω with a suitable smooth boundary, and $n|_{\mathbf{R}^d \setminus \Omega} = 1$. Given an incident wave $u^i(x) = e^{i\sqrt{\lambda}x \cdot \omega}$ with an incident direction $\omega \in S^{d-1}$ and energy $\lambda > 0$, the scattered wave u^s is described by the difference between the total wave u and the incident wave u^i where u is the solution of (1.4) satisfying the following asymptotic relation : as $|x| \rightarrow \infty$

$$(1.5) \quad u(x) \simeq e^{i\sqrt{\lambda}x \cdot \omega} + C(\lambda)|x|^{-(d-1)/2}e^{i\sqrt{\lambda}|x|}A(\lambda; \omega, \theta), \quad \theta = x/|x|.$$

Here the second term on the right-hand side is the spherical wave scattered to the direction θ . The function $A(\lambda; \omega, \theta)$ is the scattering amplitude. The *S-matrix* is given by $S(\lambda) = 1 - 2\pi i A(\lambda)$ where $A(\lambda)$ is an integral operator with the kernel $A(\lambda; \omega, \theta)$. Then the S-matrix is unitary operator on $L^2(S^{d-1})$. If there exists a non zero function $\phi \in L^2(S^{d-1})$ such that $S(\lambda)\phi = \phi$ i.e. $A(\lambda)\phi = 0$, we call $\lambda > 0$ a *non-scattering energy*. If $\lambda > 0$ is a non-scattering energy, we have that $u - u^i$ vanishes outside of Ω from the Rellich type uniqueness theorem (see [13] and [19]). Hence we can reduce to the ITE problem

$$(1.6) \quad (-\Delta - \lambda n)v = 0 \quad \text{in } \Omega,$$

$$(1.7) \quad (-\Delta - \lambda)w = 0 \quad \text{in } \Omega,$$

$$(1.8) \quad v = w, \quad \partial_\nu v = \partial_\nu w \quad \text{on } \partial\Omega,$$

with $v = u$ and $w = u^i$. If $\lambda > 0$ is a non-scattering energy, λ is also an ITE of the system (1.6)-(1.8). ITE problems were introduced in [9] and [5] in the above view point. For the Schrödinger equation $(-\Delta + V - \lambda)u = 0$ with a compactly supported potential V which satisfies $V(x) \geq \delta > 0$ in $\text{supp} V$, we can state the ITE problem similarly. Recently, [20] introduced the ITE problems on unbounded domains, considering perturbations which decrease exponentially at infinity.

The system (1.6)-(1.8) is some kind of non-self-adjoint problem. Moreover, we can construct a bilinear form associated with this system, but generally this bilinear form is not coercive. Note that the *T-coercivity* approach is valid for some anisotropic cases i.e. $-\Delta$ is replaced by $-\nabla \cdot A \nabla$ where A is a strictly positive symmetric matrix valued function and $A \neq Id$. For the *T-coercivity* approach on this case, see [3]. Another common approach is to reduce an ITE problem to an equivalent forth-order equation. For (1.6)-(1.8), we can reduce to

$$(1.9) \quad (\Delta + \lambda n) \frac{1}{n-1} (\Delta + \lambda) \psi = 0, \quad \psi = w - v \in H_0^2(\Omega),$$

which is formulated as the variational form

$$(1.10) \quad \int_{\Omega} \frac{1}{n-1} (\Delta \psi + \lambda \psi) (\Delta \bar{\phi} + \lambda n \bar{\phi}) dx = 0,$$

for any $\phi \in H_0^2(\Omega)$. There are also many works on this approach for acoustic wave equations and Schrödinger equations. For more history, technical information and references on ITE problems, we recommend the recent survey by Cakoni and Haddar [4].

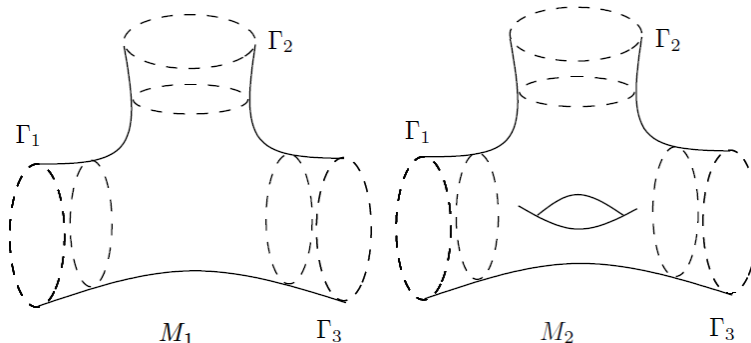


FIGURE 1. Examples of M_1 and M_2 with common boundary $\Gamma = \bigcup_{j=1}^3 \Gamma_j$.

In this paper, we slightly generalize the “isotropic type” ITE problem as a system on two domains M_1 and M_2 . Since we do not assume that M_1 and M_2 are diffeomorphic, it is difficult to use the forth-order equation approach (see Figure 1). Moreover, in view of assumptions (A-1) and (A-2) which is added in §2.3, the ITE problem is not elliptic, and we can not construct a suitable isomorphism T such that the system (1.1)-(1.3) is T -coercive. Therefore, neither the variational formulation approach nor the T -coercivity approach are valid for the proof of discreteness of ITEs in our case. Then we adopt arguments by Lakshitanov and Vainberg [11] and [10] in the present paper. They considered isotropic cases in [11] and anisotropic cases in [10] in bounded domains in \mathbf{R}^d . The approach in [11] is based on methods of elliptic pseudo differential operators on the boundary and its application to the Dirichlet-to-Neumann (D-N) map.

We should also mention about [21] and [12]. Recently, they proved the Weyl’s asymptotics including complex ITEs and evaluated ITE-free regions in the complex plane under various conditions. They used the semi-classical analysis for the D-N map associated with an operator of the form $-n(x)^{-1}\nabla \cdot c(x)\nabla$ where n, c are smooth and positive valued function on a bounded domain $\Omega \subset \mathbf{R}^d$.

Even for manifolds, these D-N map approaches are valid. More precisely, in this paper, we construct the Poisson operator and the associated D-N map as elliptic singular integro-differential operators and we can compute exactly their symbols. Using the ellipticity of the D-N map and the analytic Fredholm theory, we can prove the discreteness of the set of ITEs. Moreover, we consider a Weyl type lower bound of the number of positive ITEs except for a small neighborhood of the origin.

A case which we can use the T -coercivity approach will be studied in the forthcoming paper [15].

1.3. Plan of the paper. The plan of the paper is as follows. As has been mentioned above, the main stream of our argument is based on Lakshitanov and Vainberg [11]. In §2, we recall some basic properties of the Dirichlet-to-Neumann map. For our purpose, we need to study about residues and regular parts of the Dirichlet-to-Neumann map near its poles. The relation between ITEs and non-trivial kernels of the difference of Dirichlet-to-Neumann maps is also introduced here. Finally, we construct an approximate solution of the Dirichlet boundary value problems as a

singular-integro differential operator, and we compute the symbol of the Dirichlet-to-Neumann map. Under the preparation of §2, we prove our main results in §3. We use of the analytic Fredholm theory, the parameter ellipticity of pseudo differential operators and Weyl type asymptotic estimates for the number of Dirichlet eigenvalues on compact manifolds. Our main results are Theorem 3.4 for the discreteness of ITEs and Theorem 3.12 for the lower bound of the number of ITEs in (α, ∞) with sufficiently small $\alpha > 0$.

1.4. Notation. We use the following notations. We put $\mathbf{R}_{\geq 0} := [0, \infty)$ and $\mathbf{R}_{> 0} := (0, \infty)$. For the Riemannian metric $g_k = (g_{k,ij})$ of M_k , $\sqrt{g_k}$ and (g_k^{ij}) denote $\sqrt{\det g_k}$ and g_k^{-1} , respectively. $dV_k(x) := \sqrt{g_k}dx$ and $dS(x)$ denote the volume element on M_k and the surface element on Γ induced by $dV_k(x)$, respectively. We often write them as dV_k and dS omitting (x) . Letting $x = (x_1, \dots, x_d)$ be a local coordinate of M_k , ∂_j or ∂_{x_j} denote $\partial/\partial x_j$. For ξ , we use the similar manner. For a multiple index $\alpha = (\alpha_1, \dots, \alpha_d)$, we write $\partial_x^\alpha = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$. We often compute some kind of symbols $p(x, \xi)$. For short, we use the symbol $p(x, i\partial_x)$ which means that each ξ_j replaced by $i\partial_{x_j}$. Similarly, when we write $p(-i\partial_\xi, \xi)$, each x_j is replaced by $-i\partial_{\xi_j}$. ∂_{ν_k} denotes the outward normal derivative on Γ associated with M_k . For a strictly positive valued function $\eta \in L^\infty(M_k)$, $L^2(M_k, \eta dV_k)$ is the L^2 space on M_k with the inner product $(u, v)_{L^2(M_k, \eta dV_k)} = (\eta u, v)_{L^2(M_k)}$.

2. DIRICHLET-TO-NEUMANN MAP

2.1. Dirichlet-to-Neumann map. Here we consider the following Dirichlet problems :

$$(2.1) \quad (-\Delta_{g_k} - \lambda n_k)u_k = 0 \quad \text{in } M_k, \quad u_k = f \quad \text{on } \Gamma,$$

for $k = 1, 2$. We define the Dirichlet-to-Neumann (D-N) map $\Lambda_k(\lambda)$ by

$$(2.2) \quad \Lambda_k(\lambda)f = \partial_{\nu_k} u_k \quad \text{on } \Gamma,$$

where u_k is a solution of (2.1).

In the following, we call λ a Dirichlet eigenvalue if there exists a non-trivial solution of the equation

$$(2.3) \quad (-\Delta_{g_k} - \lambda n_k)u_k = 0, \quad \text{in } M_k, \quad u_k = 0 \quad \text{on } \Gamma.$$

In fact, (2.3) is equivalent to

$$(2.4) \quad (-n_k^{-1}\Delta_{g_k} - \lambda)u_k = 0, \quad \text{in } M_k, \quad u_k = 0 \quad \text{on } \Gamma,$$

which is an eigenvalue problem of the second-order self-adjoint elliptic operator $L_k = -n_k^{-1}\Delta_{g_k}$ in $L^2(M_k, n_k dV_k)$ with the Dirichlet boundary condition. Then its eigenvalues form an increasing sequence $0 < \lambda_{k,1} \leq \lambda_{k,2} \leq \dots$, satisfying the Weyl's asymptotics which we derive in §3. The corresponding eigenfunctions $\phi_{k,j}$ can be chosen so that $\{\phi_{k,j}\}$ is an orthonormal basis in $L^2(M_k, n_k dV_k)$. We denote the set of Dirichlet eigenvalues by $\{\lambda_{k,j}\} := \{\lambda_{k,j}\}_{j=1}^\infty$. For $\lambda \notin \{\lambda_{k,j}\}$, the D-N map $\Lambda_k(\lambda)$ is well-defined and extends uniquely as a continuous operator $\Lambda_k(\lambda) : H^{3/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$.

Let $\mathcal{E}_{k,j} \subset \mathbf{Z}_+$ such that $\bigcup_{j=1}^\infty \mathcal{E}_{k,j} = \mathbf{Z}_+$, and i_1 and i_2 belong to the same set $\mathcal{E}_{k,j}$ if and only if $\lambda_{k,i_1} = \lambda_{k,i_2}$. We denote eigenvalues corresponding $\mathcal{E}_{k,j}$ by $\lambda_{k,(j)}$. $\mathcal{L}(\lambda_{k,i})$ means the set $\mathcal{E}_{k,j}$ with $\lambda_{k,(j)} = \lambda_{k,i}$.

Proposition 2.1. $\Lambda_k(\lambda)$ is meromorphic with respect to $\lambda \in \mathbf{C}$ and has first order poles at each $\lambda \in \{\lambda_{k,j}\}$. Moreover, $\Lambda_k(\lambda)$ has the following representations :

(1) For $x \in \Gamma$ and $f \in H^{3/2}(\Gamma)$, we have

$$(2.5) \quad \Lambda_k(\lambda)f(x) = - \int_{\Gamma} \sum_{j=1}^{\infty} \frac{\partial_{\nu_k(x)} \phi_{k,j}(x) \partial_{\nu_k(y)} \phi_{k,j}(y)}{\lambda_{k,j} - \lambda} f(y) dS(y).$$

(2) In a neighborhood of $\lambda_{k,j}$, we have

$$(2.6) \quad \Lambda_k(\lambda) = \frac{Q_{k,\mathcal{L}(\lambda_{k,j})}}{\lambda_{k,j} - \lambda} + H_k(\lambda),$$

where $Q_{k,\mathcal{L}(\lambda_{k,j})}$ is the residue of $\Lambda_k(\lambda)$ at $\lambda = \lambda_{k,j}$ given by

$$(2.7) \quad Q_{k,\mathcal{L}(\lambda_{k,j})}f = - \sum_{i \in \mathcal{L}(\lambda_{k,j})} \int_{\Gamma} \partial_{\nu_k(y)} \phi_{k,i}(y) f(y) dS(y) \partial_{\nu_k} \phi_{k,i},$$

and $H_k(\lambda) : H^{3/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$ is analytic in a neighborhood of $\lambda_{k,j}$.

Proof. We can follow the argument of §4.1.12 in [8]. Let $E_k \in H^2(M_k)$ be an extension of f into M_k satisfying $E_k|_{\Gamma} = f$ and $\|E_k\|_{H^2(M_k)} \leq C\|f\|_{H^{3/2}(\Gamma)}$ for some constants $C > 0$, and u_k is in (2.1). Then we have

$$(-n_k^{-1}\Delta_{g_k} - \lambda)(u_k - E_k) = -(-n_k^{-1}\Delta_{g_k} - \lambda)E_k.$$

Since $R_k(\lambda) := (-n_k^{-1}\Delta_{g_k} - \lambda)^{-1}$ is a meromorphic operator valued function with first order poles only at $\lambda \in \{\lambda_{k,j}\}$, $u_k = E_k - R_k(\lambda)(-n_k^{-1}\Delta_{g_k} - \lambda)E_k$ is also a meromorphic $H^2(M_k)$ -valued function with first order poles only at $\lambda \in \{\lambda_{k,j}\}$.

Next we prove (2.5). Integrating by parts, we compute the Fourier coefficients of u_k with respect to the real-valued eigenfunction $\phi_{k,j}$:

$$(2.8) \quad (u_k, \phi_{k,j})_{L^2(M_k, n_k dV_k)} = - \int_{\Gamma} \frac{\partial_{\nu_k(y)} \phi_{k,j}(y)}{\lambda_{k,j} - \lambda} f(y) dS(y).$$

From this formula and the outward normal derivative of u_k , $\Lambda_k(\lambda)$ satisfies (2.5).

Finally we verify (2.6) and (2.7). Let $P_{k,j} : L^2(M_k, n_k dV_k) \rightarrow L^2(M_k, n_k dV_k)$ be the projection to the eigenspace corresponding to $\lambda_{k,j}$ i.e.

$$P_{k,j}v = \sum_{i \in \mathcal{L}(\lambda_{k,j})} (v, \phi_{k,i})_{L^2(M_k, n_k dV_k)} \phi_{k,i}, \quad v \in L^2(M_k, n_k dV_k).$$

In view of (2.8), we have

$$P_{k,j}u_k = - \frac{1}{\lambda_{k,j} - \lambda} \sum_{i \in \mathcal{L}(\lambda_{k,j})} \int_{\Gamma} \partial_{\nu_k(y)} \phi_{k,i}(y) f(y) dS(y) \phi_{k,i},$$

and this implies (2.7). Moreover,

$$(1 - P_{k,j})u_k = - \sum_{i \notin \mathcal{L}(\lambda_{k,j})} \frac{1}{\lambda_{k,i} - \lambda} \int_{\Gamma} \partial_{\nu_k(y)} \phi_{k,i}(y) f(y) dS(y) \phi_{k,i},$$

is analytic with respect to λ in a neighborhood of $\lambda_{k,j}$. Putting $H_k(\lambda)f = \partial_{\nu_k}((1 - P_{k,j})u_k)$ on Γ , we have the proposition. \square

Remark. The formula (2.7) means that the range of $Q_{k,\mathcal{L}(\lambda_{k,j})}$ is a finite dimensional subspace spanned by $\partial_{\nu_k} \phi_{k,i}$ for $i \in \mathcal{L}(\lambda_{k,j})$. Note that $\partial_{\nu_k} \phi_{k,i}$ for all $i \in \mathcal{L}(\lambda_{k,j})$ are linear independent since $\phi_{k,i}$ are orthogonal basis. Hence

$\dim \text{Ran} Q_{k, \mathcal{L}(\lambda_{k,j})}$ coincides with the multiplicity of $\lambda_{k,j}$. We can see that the integral kernel of $Q_{k, \mathcal{L}(\lambda_{k,j})}$ is smooth in (x, y) by the regularity property of Dirichlet eigenfunctions.

As has been in Propositions 2.1, $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ is also meromorphic with respect to $\lambda \in \mathbf{C}$ and has first order poles at each $\lambda \in \{\lambda_{1,j}\} \cup \{\lambda_{2,j}\}$. In a neighborhood of a pole λ_0 , we have

$$(2.9) \quad \Lambda_1(\lambda) - \Lambda_2(\lambda) = \frac{Q_{\lambda_0}}{\lambda_0 - \lambda} + H_{\lambda_0}(\lambda),$$

where Q_{λ_0} and $H_{\lambda_0}(\lambda)$ have same properties of $Q_{k, \mathcal{L}(\lambda_{k,j})}$ and $H_k(\lambda)$, respectively. In the following, we define the kernel of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ by

$$(2.10) \quad \begin{aligned} & \text{Ker}(\Lambda_1(\lambda) - \Lambda_2(\lambda)) \\ &= \begin{cases} \{f \in H^{3/2}(\Gamma) ; (\Lambda_1(\lambda) - \Lambda_2(\lambda))f = 0\}, & \text{if } \lambda \text{ is not a pole,} \\ \{f \in H^{3/2}(\Gamma) ; Q_{\lambda_0}f = H_{\lambda_0}(\lambda_0)f = 0\}, & \text{if } \lambda = \lambda_0 \text{ is a pole.} \end{cases} \end{aligned}$$

Lemma 2.2. *Let $\lambda \in \{\lambda_{k,j}\}$. Then the equation (2.1) has a non trivial solution if and only if f is orthogonal to $\partial_{\nu_k} \phi_{k,j}$ in $L^2(\Gamma)$ for all $j \in \mathcal{L}(\lambda)$.*

Proof. If f is orthogonal to $\partial_{\nu_k} \phi_{k,j}$ for all $j \in \mathcal{L}(\lambda)$, there exist general solutions of the form

$$(2.11) \quad u_k = - \sum_{i \notin \mathcal{L}(\lambda)} \frac{1}{\lambda_{k,i} - \lambda} \int_{\Gamma} \partial_{\nu_k(y)} \phi_{k,i}(y) f(y) dS(y) \phi_{k,i} + \sum_{i \in \mathcal{L}(\lambda)} c_i \phi_{k,i},$$

for any $c_i \in \mathbf{C}$.

If u_k is a non trivial solution of (2.1), we have by Green's formula

$$\int_{M_k} (\Delta_{g_k} u_k \cdot \phi_{k,i} - u_k \cdot \Delta_{g_k} \phi_{k,i}) dV_k = - \int_{\Gamma} u_k \cdot \partial_{\nu_k} \phi_{k,i} dS,$$

for $i \in \mathcal{L}(\lambda)$. Since $\lambda = \lambda_{k,i}$, the left-hand side is equal to zero. Then $f = u_k|_{\Gamma}$ is orthogonal to $\partial_{\nu_k} \phi_{k,i}$. \square

The above lemma implies a unique solvability in a subspace as follows.

Corollary 2.3. *Let $\lambda = \lambda_0$ be a Dirichlet eigenvalue. We define $E_k(\lambda_0) \subset H^2(M_k)$ as the eigenspace associated by λ_0 , and $B_k(\lambda_0)$ as the subspace of $H^{3/2}(\Gamma)$ spanned by $\partial_{\nu_k} \phi_{k,j}$ for all $j \in \mathcal{L}(\lambda_0)$. We denote by $E_k(\lambda_0)^c$ and $B_k(\lambda_0)^c$ their orthogonal complements in $L^2(M_k)$ and $L^2(\Gamma)$, respectively. For any $f \in B_k(\lambda_0)^c$, there exists a unique solution $u_k \in E_k(\lambda_0)^c \cap H^2(M_k)$ of (2.1) represented by*

$$(2.12) \quad u_k = - \sum_{i \notin \mathcal{L}(\lambda_0)} \frac{1}{\lambda_{k,i} - \lambda} \int_{\Gamma} \partial_{\nu_k(y)} \phi_{k,i}(y) f(y) dS(y) \phi_{k,i}.$$

Proof. We have only to check the uniqueness. This is trivial since the equation (2.3) has only the trivial solution in $E_k(\lambda_0)^c$. \square

Now we can state the relation between ITEs and the D-N map as follows.

Lemma 2.4. (1) *Suppose $\lambda \notin \{\lambda_{1,j}\} \cap \{\lambda_{2,j}\}$. Then $\lambda \in \mathbf{C}$ is an ITE if and only if $\text{Ker}(\Lambda_1(\lambda) - \Lambda_2(\lambda)) \neq \{0\}$. The multiplicity of λ coincides with $\dim(\text{Ker}(\Lambda_1(\lambda) - \Lambda_2(\lambda)))$.*

(2) *Suppose $\lambda \in \{\lambda_{1,j}\} \cap \{\lambda_{2,j}\}$. Then $\lambda \in \mathbf{C}$ is an ITE if and only if $\text{Ker}(\Lambda_1(\lambda) -$*

$\Lambda_2(\lambda)) \neq \{0\}$ or the ranges of $Q_{1,\mathcal{L}(\lambda)}$ and $Q_{2,\mathcal{L}(\lambda)}$ have a non trivial intersection. The multiplicity of λ coincides with the sum of $\dim(\text{Ker}(\Lambda_1(\lambda) - \Lambda_2(\lambda)))$ and the dimension of the above intersection.

Proof. We first prove the assertion (1). When $\lambda \notin \{\lambda_{1,j}\} \cup \{\lambda_{2,j}\}$, this lemma is a direct consequence of the definition of ITEs. We have only to show for $\lambda \in \{\lambda_{1,j}\} \setminus \{\lambda_{2,j}\}$. For $0 \neq f \in \text{Ker}(\Lambda_1(\lambda) - \Lambda_2(\lambda))$, we have $Q_{1,\mathcal{L}(\lambda)}f = (H_1(\lambda) - \Lambda_2(\lambda))f = 0$. From $Q_{1,\mathcal{L}(\lambda)}f = 0$ and (2.7), we have $f \in B_1(\lambda)^c$. By Lemma 2.2 and Corollary 2.3, the following equation has a unique non trivial solution :

$$(2.13) \quad (-\Delta_{g_1} - \lambda n_1)u_1 = 0 \quad \text{in } M_1, \quad u_1 = f \quad \text{on } \Gamma.$$

On the other hand, from $\Lambda_2(\lambda)f = H_1(\lambda)f$, we have

$$(2.14) \quad (-\Delta_{g_2} - \lambda n_2)u_2 = 0 \quad \text{in } M_2, \quad u_2 = f, \quad \partial_{\nu_2}u_2 = H_1(\lambda)f \quad \text{on } \Gamma.$$

Summarizing (2.13) and (2.14) and $\partial_{\nu_1}u_1 = H_1(\lambda)f$, λ is an ITE. Conversely, if λ is an ITE, from Lemma 2.2, the equation (2.1), $k = 1$, with the condition $u_1|_{\Gamma} = f \neq 0$ must have a non trivial solution. In view of (2.7), we have $f \in B_1(\lambda)^c$, and this implies $Q_{1,\mathcal{L}(\lambda)}f = 0$. This means $\partial_{\nu_1}u_1 = \Lambda_1(\lambda)f = H_1(\lambda)f$ by using (2.6). On the other hand, $\partial_{\nu_1}u_1 = \partial_{\nu_2}u_2$ means $(H_1(\lambda) - \Lambda_2(\lambda))f = 0$. Therefore, f must be in $\text{Ker}(\Lambda_1(\lambda) - \Lambda_2(\lambda))$. We have proven the assertion (1).

For the assertion (2), we have only to show the latter case. In fact, if there exists a non trivial solution (u_1, u_2) of

$$\begin{aligned} (-\Delta_{g_1} - \lambda n_1)u_1 &= 0, \quad \text{in } M_1, \\ (-\Delta_{g_2} - \lambda n_2)u_2 &= 0, \quad \text{in } M_2, \end{aligned}$$

with $u_1|_{\Gamma} = u_2|_{\Gamma} = 0$ and $\partial_{\nu_1}u_1 = \partial_{\nu_2}u_2$ on Γ , then we have that the ranges of $Q_{1,\mathcal{L}(\lambda)}$ and $Q_{2,\mathcal{L}(\lambda)}$ have a non trivial intersection, recalling $\text{Ran}Q_{k,\mathcal{L}(\lambda)} = \text{Span}\{\partial_{\nu_k}\phi_{k,j}\}_{j \in \mathcal{L}(\lambda)}$ for $k = 1, 2$. Conversely, if the ranges of $Q_{1,\mathcal{L}(\lambda)}$ and $Q_{2,\mathcal{L}(\lambda)}$ have a non trivial intersection, then there exists a non trivial solution (u_1, u_2) of the above system with the condition $u_1|_{\Gamma} = u_2|_{\Gamma} = 0$ and $\partial_{\nu_1}u_1 = \partial_{\nu_2}u_2$ on Γ , since $\partial_{\nu_k}\phi_{k,i}$ for all $i \in \mathcal{L}(\lambda_0)$ are linear independent. Then λ is an ITE. \square

Remark. In [10], the authors call λ *singular spectral point* if λ satisfies the latter condition in the assertion (2) of Lemma 2.4.

2.2. Local regularizer. Now let us compute the symbol of the D-N map. Here we construct the local regularizer for (2.1). As in [11], we follow the argument of §2 in [18], slightly modifying it for our case.

In the following, we assume that the equation (2.1) is uniquely solvable in $H^2(M_k)$ or a suitable subspace of $L^2(M_k)$.

We take a point $x^{(0)} \in \Gamma$ and fix it. Let $V \subset \Gamma$ be a sufficiently small neighborhood of $x^{(0)}$ in Γ . There exist small open domains $U_k \subset M_k$, $k = 1, 2$, such that $\overline{U_k} \cap \Gamma = V$ and U_1 and U_2 are diffeomorphic to an open domain $\Omega \subset \mathbf{R}^d$.

We introduce local coordinates $y = (y_1, \dots, y_{d-1}, y_d)$ in Ω with the center $x^{(0)} \in V$ such that $x^{(0)} = 0$, Ω is given by $y_d > 0$, $|y| < \epsilon_0$ for a small $\epsilon_0 > 0$, the subset $\partial\Omega^0 := \{y \in \overline{\Omega} ; y_d = 0\}$ is diffeomorphic to V , and y_d is the distance between a point $y = (y_1, \dots, y_{d-1}, y_d) \in \Omega$ and $\partial\Omega^0$. Then $y = (y_1, \dots, y_d)$ are common local coordinates of U_1 and U_2 . Therefore, we have

$$(g_k^{ij}(y))_{i,j} = \begin{bmatrix} \tilde{g}_k(y') & \tilde{p}_k(y) \\ \tilde{p}_k(y) & 1 \end{bmatrix}, \quad y' = (y_1, \dots, y_{d-1}),$$

in U_k where $\tilde{g}_k(y') = (\tilde{g}_k^{ij}(y'))_{i,j}$ is a smooth, positive definite and symmetric $(d-1) \times (d-1)$ -matrix valued function, and $\tilde{p}_k(y) = {}^t(p_{k,1}(y), \dots, \tilde{p}_{k,d-1}(y))$ is a $(d-1)$ -dimensional vector valued function.

A function $F(y', y_d, \xi', \xi_d)$ with $(y', y_d), (\xi', \xi_d) \in \mathbf{R}^d$ is homogeneous of the generalized degree s if F satisfies

$$(2.15) \quad F(t^{-1}y', t^{-1}y_d, t\xi', t\xi_d) = t^s F(y', y_d, \xi', \xi_d),$$

for any $t > 0$. For $F(y_d, \xi')$, we define the homogeneity by the similar manner.

Taking the y -coordinate as above, we can rewrite $A_k = -\Delta_{g_k} - \lambda n_k$ as

$$(2.16) \quad A_k =: -\partial_d^2 - \sum_{i,j=1}^{d-1} \tilde{g}_k^{ij}(y') \partial_i \partial_j - 2 \sum_{i=1}^d \tilde{p}_{k,i}(y) \partial_i \partial_d - \sum_{i=1}^d \tilde{h}_{k,i}(y) \partial_i - \lambda n_k(y),$$

in U_k with $\tilde{h}_{k,i}(y) = (\sqrt{g_k})^{-1} \sum_{j=1}^d \partial_j (\sqrt{g_k} g_k^{ij})$. Note that $\tilde{g}_k^{ij}(y')$, $\tilde{p}_{k,i}(y)$ and $\tilde{h}_{k,i}(y)$ are defined by $g_k(y)$. In view of the assumption (A-1), we have in y -coordinates that $\tilde{g}_1^{ij}(y') = \tilde{g}_2^{ij}(y')$, $\tilde{p}_{1,i}(y)|_{y_d=0} = \tilde{p}_{2,i}(y)|_{y_d=0} = 0$.

The symbol of A_k is given by

$$(2.17) \quad \begin{aligned} & A_k(\lambda; y', y_d, \xi', \xi_d) \\ &= \xi_d^2 + \sum_{i,j=1}^{d-1} \tilde{g}_k^{ij}(y') \xi_i \xi_j + 2 \sum_{i=1}^d \tilde{p}_{k,i}(y) \xi_i \xi_d - i \sum_{i=1}^{d-1} \tilde{h}_{k,i}(y) \xi_i - \lambda n_k(y). \end{aligned}$$

In the following, let $N > 0$ be a sufficiently large integer. Now we take $z = (z', 0) \in \partial\Omega^0$ arbitrarily and fix it. Using the Taylor series of $\tilde{g}_k^{ij}(y')$, $\tilde{p}_{k,i}(y)$, $\tilde{h}_{k,i}(y)$ and $n_k(y)$ with respect to y centered at $(z', 0) \in \partial\Omega^0$, we can expand the symbol $A_k(y', y_d, \xi', \xi_d)$ of A_k as the sum of following terms :

$$(2.18) \quad \xi_d^2 + \sum_{i,j=1}^{d-1} \tilde{g}_k^{ij}(z') \xi_i \xi_j,$$

$$(2.19) \quad \begin{aligned} & \sum_{i,j=1}^{d-1} \nabla_{y'} \tilde{g}_k^{ij}(z') \cdot (y' - z') \xi_i \xi_j + i \sum_{i=1}^d \tilde{h}_{k,i}(z', 0) \xi_i \\ & + 2 \sum_{i=1}^{d-1} (\nabla_{y'} \tilde{p}_{k,i}(z', 0) \cdot (y' - z') + \partial_d \tilde{p}_{k,i}(z', 0) y_d) \xi_i \xi_d, \end{aligned}$$

and

$$(2.20) \quad \begin{aligned} & \sum_{i,j=1}^{d-1} \sum_{|\alpha'|=m} \frac{\partial_{y'}^{\alpha'} \tilde{g}_k^{ij}(z')}{\alpha'!} (y' - z')^{\alpha'} \xi_i \xi_j + 2 \sum_{i=1}^d \sum_{|\alpha|=m} \frac{\partial_y^{\alpha} \tilde{p}_{k,i}(z', 0)}{\alpha!} (y' - z')^{\alpha'} y_d^{\alpha_d} \xi_i \xi_d \\ & + i \sum_{i=1}^d \sum_{|\alpha|=m-1} \frac{\partial_y^{\alpha} \tilde{h}_{k,i}(z', 0)}{\alpha!} (y' - z')^{\alpha'} y_d^{\alpha_d} \xi_i - \lambda \sum_{|\alpha|=m-2} \frac{\partial_y^{\alpha} n_k(z', 0)}{\alpha!} (y' - z')^{\alpha'} y_d^{\alpha_d}, \end{aligned}$$

for $2 \leq m \leq N$ with the remainder term which has zero of order $N - 1$ at $y' = 0$ or $(y', y_d) = (0, 0)$. We rewrite the sum of (2.18)-(2.20) and the remainder term as

$$(2.21) \quad \begin{aligned} A_k(\lambda; y', y_d, \xi', \xi_d) &= A_{k,0}(z'; \xi', \xi_d) + A_{k,1}(z'; y' - z', y_d, \xi', \xi_d) \\ &+ \sum_{m=2}^N A_{k,m}(\lambda, z'; y' - z', y_d, \xi', \xi_d) + A'_{k,N+1}(\lambda, z'; y' - z', y_d, \xi', \xi_d). \end{aligned}$$

Then each $A_{k,m}$ is a homogeneous polynomial in $y' - z', y_d, \xi', \xi_d$ of generalized degree $2 - m$. In particular, $A_{k,0}$ is the principal symbol of A_k . $A'_{k,N+1}$ vanishes at $(z', 0)$ and the order of the zero is $N - 1$.

In the following arguments, we put

$$(2.22) \quad |\xi'|_{\Gamma}^2 := \sum_{i,j=1}^{d-1} \tilde{g}_k^{ij}(y') \xi_i \xi_j.$$

We define the following differential operators :

$$(2.23) \quad \tilde{A}_{k,0} = A_{k,0}(z'; \xi', i\partial_d) = -\partial_d^2 + |\xi'|_{\Gamma}^2,$$

$$(2.24) \quad \tilde{A}_{k,1} = A_{k,1}(z'; -i\partial_{\xi'}, y_d, \xi', i\partial_d),$$

and

$$(2.25) \quad \tilde{A}_{k,m} = A_{k,m}(\lambda, z'; -i\partial_{\xi'}, y_d, \xi', i\partial_d), \quad m \geq 2.$$

Proposition 2.5. *Let $F(y_d, \xi')$ be a smooth function and homogeneous of the generalized degree s with respect to y_d and ξ' . Then we have that $\tilde{A}_{k,m}F$ is the homogeneous of the generalized degree $2 - m + s$ with respect to y_d and ξ' .*

Proof. Note that $F(y_d, \xi') = |\xi'|^s F(|\xi'| y_d, \xi' / |\xi'|)$. Then we can show that $\partial_d F$ and $\partial_{\xi_j} F$ are homogeneous of generalized degree $s + 1$ and $s - 1$, respectively. \square

Now let us construct an approximate solution of (2.1).

Lemma 2.6. *Suppose $|\xi'|_{\Gamma} \neq 0$. The system of second order ordinary differential equations*

$$(2.26) \quad \tilde{A}_{k,0} E_{k,0}(z'; y_d, \xi') = 0,$$

$$(2.27) \quad \tilde{A}_{k,0} E_{k,1}(z'; y_d, \xi') = -\tilde{A}_{k,1} E_{k,0}(z'; y_d, \xi'),$$

...

$$(2.28) \quad \tilde{A}_{k,0} E_{k,m}(z'; y_d, \xi') = -\sum_{n=1}^m \tilde{A}_{k,n} E_{k,m-n}(z'; y_d, \xi'),$$

has a unique solution $\{E_{k,m}\}_{m=0,1,2,\dots}$ such that each $E_{k,m}$ converges to zero as $y_d \rightarrow \infty$ and satisfies

$$E_{k,0}|_{y_d=0} = 1, \quad E_{k,m}|_{y_d=0} = 0, \quad m \geq 1.$$

In particular, we have $E_{k,0}(z'; y_d, \xi') = e^{-|\xi'|_{\Gamma} y_d}$. Each solution $E_{k,m}$ is smooth and homogeneous with respect to y_d and ξ' of generalized degree $-m$. (For $m \geq 2$, each $E_{k,m}$ depends also on λ . We omit λ in the notation.)

Proof. Since $\tilde{A}_{k,0} = -\partial_d^2 + |\xi'|_\Gamma^2$, we have $E_{k,0}(z'; y_d, \xi') = e^{-|\xi'|_\Gamma y_d}$. Obviously, $E_{k,0}$ is homogeneous of the generalized degree 0. Let us consider the equation

$$(2.29) \quad (-\partial_d^2 + |\xi'|_\Gamma^2)v = p \quad \text{on } (0, \infty),$$

for $v(y_d, \xi')$ and $p(y_d, \xi')$ with $v(0, \xi') = 0$, $v(y_d, \xi') \rightarrow 0$ as $y_d \rightarrow \infty$. Here we assume that $p(y_d, \xi')$ decays exponentially as $y_d \rightarrow \infty$ and is homogeneous of the generalized degree s . Extending v and p to be zero in $-\infty < y_d < 0$, we have

$$v(y_d, \xi') = \frac{1}{2|\xi'|_\Gamma} \left(\int_0^{y_d} e^{-|\xi'|_\Gamma(y_d-\eta)} p(\eta, \xi') d\eta + \int_{y_d}^\infty e^{-|\xi'|_\Gamma(\eta-y_d)} p(\eta, \xi') d\eta \right).$$

Then, putting $\tau = t\eta$, we have

$$\begin{aligned} & v(t^{-1}y_d, t\xi') \\ &= \frac{t^{s-2}}{2|\xi'|_\Gamma} \left(\int_0^{y_d} e^{-|\xi'|_\Gamma(y_d-\tau)} p(\tau, \xi') d\tau + \int_{y_d}^\infty e^{-|\xi'|_\Gamma(\tau-y_d)} p(\tau, \xi') d\tau \right) \\ &= t^{s-2}v(y_d, \xi'), \end{aligned}$$

which shows that v is homogeneous of the generalized degree $s-2$ with respect to y_d and ξ' . In view of Proposition 2.5, we have $\tilde{A}_{k,1}E_{k,0}$ is homogeneous of the generalized degree 1. Therefore, we obtain $E_{k,1}$ is homogeneous of the generalized degree -1 . Repeating the similar argument inductively, we can show that $E_{k,m}$ is homogeneous of the generalized degree $-m$. \square

Let $\beta(\xi') \in C^\infty(\mathbf{R}^{d-1})$ vanish in a neighborhood of $\xi' = 0$, and be equal to one outside a large neighborhood of $\xi' = 0$. Taking $\psi \in H^{3/2}(\partial\Omega^0)$ with a compact support in $\partial\Omega^0$, we define for $y' \in \partial\Omega^0$

$$(2.30) \quad \begin{aligned} & (Q_{k,m}\psi)(z'; y', y_d) \\ &= (2\pi)^{-(d-1)} \int e^{iy' \cdot \xi'} \beta(\xi') E_{k,m}(z'; y_d, \xi') \int e^{-iw' \cdot \xi'} \psi(w') dw' d\xi', \end{aligned}$$

and we put

$$(2.31) \quad R_{k,N} = \sum_{m=0}^N Q_{k,m}.$$

Letting

$$(2.32) \quad q_{k,m}(z'; y', y_d) = (2\pi)^{-(d-1)} \int e^{iy' \cdot \xi'} \beta(\xi') E_{k,m}(z'; y_d, \xi') d\xi',$$

we have that $q_{k,m}$ is a distribution in \mathcal{S}' , and

$$(2.33) \quad (Q_{k,m}\psi)(z'; y', y_d) = \int q_{k,m}(z'; y' - w', y_d) \psi(w') dw',$$

$$(2.34) \quad (R_{k,N}\psi)(z'; y', y_d) = \int r_{k,N}(z'; y' - w', y_d) \psi(w') dw',$$

with

$$r_{k,N}(z'; y' - w', y_d) = \sum_{m=0}^N q_{k,m}(z'; y' - w', y_d).$$

We represent A_k in the form

$$A_k = A_{k,0}(z'; i\partial_{y'}, i\partial_d) + A_{k,1}(z'; y' - z', y_d, i\partial_{y'}, i\partial_d) \\ + \sum_{m=2}^N A_{k,m}(\lambda, z'; y' - z', y_d, i\partial_{y'}, i\partial_d) + A'_{k,N+1}(\lambda, z'; y' - z', y_d, i\partial_{y'}, i\partial_d).$$

In the following, we consider

$$(2.35) \quad A_k r_{k,N} = \sum_{J=0}^N \sum_{l+m=J} A_{k,l} q_{k,m} + \sum_{J=N+1}^{2N} \sum_{l,m \leq N, l+m=J} A_{k,l} q_{k,m} + A'_{k,N+1} r_{k,N}.$$

Lemma 2.7. *Let l , m and N be sufficiently large. We have $A_{k,l} q_{k,m} \in H^\gamma(\Omega)$ and $A'_{k,N+1} r_{k,N} \in H^{\gamma'}(\Omega)$ where $\gamma = O(l+m)$ and $\gamma' = O(N)$.*

Proof. Note that $A_{k,l}(\lambda, z'; y' - z', y_d, i\partial_{y'}, i\partial_d)$ and $A'_{k,N+1}(\lambda, z'; y' - z', y_d, i\partial_{y'}, i\partial_d)$ are operators which are given by sums of terms like $(y' - z')^{\alpha'} y_d^{\alpha_d} \partial_{y'}^{\beta'} \partial_d^{\beta_d}$ up to a smooth function with $-|\alpha'| - \alpha_d + |\beta'| + \beta_d = 2 - l$ or $2 - (N+1)$ and $|\beta'| + \beta_d \leq 2$. In view of Proposition 2.5, it is sufficient to show

$$(2.36) \quad (y')^{\alpha'} y_d^{\alpha_d} q_{k,m}(z; y', y_d) \in H^\gamma(\Omega),$$

since the derivative $\partial_{y'}^{\beta'} \partial_d^{\beta_d}$ is order zero, one or two.

Now we have

$$(y')^{\alpha'} y_d^{\alpha_d} q_{k,m}(z; y', y_d) = i^{|\alpha'|} (2\pi)^{-(d-1)} \int e^{iy' \cdot \xi'} \partial_{\xi'}^{\alpha'} (y_d^{\alpha_d} \beta(\xi') |\xi'|^{-m} E_{k,m}(z'; |\xi'| y_d, \xi' / |\xi'|)) d\xi'.$$

Since $y_d^{\alpha_d} |\xi'|^{-m} E_{k,m}(z'; |\xi'| y_d, \xi' / |\xi'|)$ is homogeneous of the generalized degree $-m - \alpha_d$, using proposition 2.5, we have

$$\left| \partial_{\xi'}^{\alpha'} (y_d^{\alpha_d} \beta(\xi') E_{k,m}(z'; y_d, \xi')) \right| \leq C_{m,\alpha} (1 + |\xi'|)^{-m - |\alpha'| - \alpha_d},$$

which implies (2.36). \square

Theorem 2.8. *Let $N > 1$ be sufficiently large. We have that $R_{k,N}$ is local regularizer for (2.1) i.e.*

$$(2.37) \quad A_k R_{k,N} \psi \in H^s(\Omega), \quad R_{k,N} \psi|_{y_d=0} - \psi \in C^\infty(\partial\Omega^0),$$

for $\psi \in H^{3/2}(\partial\Omega^0)$ which has a compact support in $\partial\Omega^0$ and $s = O(N)$.

Proof. Note that

$$(2.38) \quad A_{k,l}(\lambda, z; y' - z', y_d, i\partial_{y'}, i\partial_d) q_{k,m}(z'; y' - w', y_d) \\ = (2\pi)^{-(d-1)} \int e^{i(y' - w') \cdot \xi'} \tilde{A}_{k,l}(\beta(\xi') E_{k,m}(z'; y_d, \xi')) d\xi'.$$

Summing up both sides of (2.26)-(2.28), we have

$$(2.39) \quad \sum_{J=0}^N \sum_{l+m=J} \tilde{A}_{k,l} E_{k,m}(z'; y_d, \xi') = 0.$$

In view of Lemma 2.7 and (2.35), we have that (2.38) and (2.39) imply $A_k R_{k,N} \psi \in H^s(\Omega)$ for $s = O(N)$.

We have that

$$\begin{aligned} & R_{k,N}\psi(y', y_d) - \psi(y') \\ &= (2\pi)^{-(d-1)} \iint e^{i(y'-w')\cdot\xi'} \left(\sum_{m=0}^N \beta(\xi') E_{k,m}(z'; y_d, \xi') - 1 \right) \psi(w') d\xi' dw' \\ &\rightarrow (2\pi)^{-(d-1)} \iint e^{i(y'-w')\cdot\xi'} (\beta(\xi') - 1) \psi(w') d\xi' dw', \end{aligned}$$

as $y_d \rightarrow 0$. Since $\beta(\xi') - 1 \in C_0^\infty(\mathbf{R}^{d-1})$, we have $R_{k,N}\psi|_{y_d=0} - \psi(y') \in C^\infty(\partial\Omega^0)$. \square

Remark. The formal sum

$$(R_k\psi)(z'; y', y_d) = \int \sum_{m=0}^{\infty} q_{k,m}(z'; y' - w', y_d) \psi(w') dw',$$

is a singular integro-differential operator (see [17]). In general, a linear operator P on a d -dimensional compact manifold M is a singular integro-differential operator of order l if there exist homogeneous functions $p_j(x, \xi) \in C^\infty(M, \mathbf{R}^d/\{0\})$ in ξ with homogeneous degree $l - j$ such that for a function u with support in a local coordinate neighborhood $U \subset M$,

$$Pu(x) = (2\pi)^{-d} \iint e^{i(x-y)\cdot\xi} \beta(\xi) \sum_{j=0}^N p_j(x, \xi) u(y) dy d\xi + T_{N+1}u, \quad x \in U,$$

where $\beta \in C^\infty(\mathbf{R}^d)$ is an arbitrary function which satisfies $\beta(\xi) = 0$ for $|\xi| \leq 1$ and $\beta(\xi) = 1$ for $|\xi| \geq 2$, and T_{N+1} is an operator which increases the smoothness i.e. $H^s(M) \rightarrow H^{s+O(N)}(M)$ for any $s \in \mathbf{R}$. The principal symbol of P is $p_0(x, \xi)$ and the full symbol of P is the formal sum $\sum_j p_j(x, \xi)$. Then the ellipticity of P is defined by $p_0(x, \xi) \neq 0$ for all $\xi \neq 0$. Here this means that we can construct the parametrix of P (see [7]). Therefore, if P is an elliptic singular integro-differential operator, P is Fredholm.

Since we have $\partial_{\nu_k} = -\partial_d$ in y -coordinates, we can show the following fact. As a consequence of Corollary 2.3 and Theorem 2.8. See also Lemma 11 and Theorem 14 in [18].

Corollary 2.9. (1) When λ is not a pole of $\Lambda_k(\lambda)$, $\Lambda_k(\lambda)$ is a singular integro-differential operator on $H^{3/2}(\Gamma)$ with the full symbol given by the following asymptotic series :

$$(2.40) \quad \Lambda_k(\lambda; y', \xi') = - \sum_{m=0}^{\infty} \partial_d E_{k,m}(y'; y_d, \xi') \Big|_{y_d=0}, \quad y' \in \partial\Omega^0.$$

(2) When $\lambda = \lambda_0$ is a pole of $\Lambda_k(\lambda)$, the regular part $H_k(\lambda)$ of $\Lambda_k(\lambda)$ at λ_0 is a singular integro-differential operator on $B_k(\lambda_0)^c$ with the full symbol given by (2.40).

2.3. Principal symbol of the D-N map. We compute the principal symbol of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$. In the following, we denote higher order normal derivatives on Γ associated with M_k by

$$(2.41) \quad \partial_{\nu_k}^m := \partial_{\nu_k} \circ \partial_{\nu_k}^{m-1}, \quad m \geq 2, \quad \partial_{\nu_k}^1 = \partial_{\nu_k}.$$

In y -coordinates, we can locally represent $\partial_{\nu_k} = (-1)^m \partial_d^m$. Under the assumption (A-1), we additionally assume on Γ that

(A-2) The metrics g_1 , g_2 and the indices of refraction n_1 , n_2 satisfy one of following two cases :

(A-2-1) For all $x \in \Gamma$, $\partial_{\nu_1}^m g_1^{ij}(x) = \partial_{\nu_2}^m g_2^{ij}(x)$ for $m \leq 2$, $i, j = 1, \dots, d$, and $n_1(x) \neq n_2(x)$,

or

(A-2-2) For all $x \in \Gamma$, $\partial_{\nu_1}^m g_1^{ij}(x) = \partial_{\nu_2}^m g_2^{ij}(x)$ for $m \leq 3$, $i, j = 1, \dots, d$, and $n_1(x) = n_2(x)$, $\partial_{\nu_1} n_1(x) \neq \partial_{\nu_2} n_2(x)$.

Note that, under the assumptions (A-1) with (A-2-1) or (A-2-2), we can see $\tilde{A}_{1,m} = \tilde{A}_{2,m}$ for $m \leq 1$ or $m \leq 2$, respectively.

When $\lambda = \lambda_0$ is a pole of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$, we define a subspace $B(\lambda_0)$ of $H^{3/2}(\Gamma)$ by $B(\lambda_0) = \tilde{B}_1(\lambda_0) \cup \tilde{B}_2(\lambda_0)$ where $\tilde{B}_k(\lambda_0) = B_k(\lambda_0)$ if λ_0 is a Dirichlet eigenvalue of $-\Delta_{g_k} - \lambda n_k$, and $\tilde{B}_k(\lambda_0) = \emptyset$ if otherwise. We denote by $B(\lambda_0)^c$ the orthogonal complement of $B(\lambda_0)$ in $L^2(\Gamma)$.

When $\lambda = \lambda_0$ is a pole of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$, we call $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ Fredholm if its regular part $H_{\lambda_0}(\lambda)$ is Fredholm.

Lemma 2.10. *In the following, we suppose $\lambda \neq 0$.*

(1) *Let λ be not a pole of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$. For the case (A-2-1), we have $\Lambda_1(\lambda) - \Lambda_2(\lambda) : H^{3/2}(\Gamma) \rightarrow H^{5/2}(\Gamma)$ is an elliptic singular integro-differential operator with the principal symbol*

$$(2.42) \quad -\frac{\lambda(n_1(x) - n_2(x))}{2|\xi'|_{\Gamma}}, \quad x \in \Gamma, \quad \xi' \in \mathbf{R}^{d-1}.$$

(2) *Let λ be not a pole of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$. For the case (A-2-2), we have $\Lambda_1(\lambda) - \Lambda_2(\lambda) : H^{3/2}(\Gamma) \rightarrow H^{7/2}(\Gamma)$ is an elliptic singular integro-differential operator with the principal symbol*

$$(2.43) \quad \frac{\lambda(\partial_{\nu_1} n_1(x) - \partial_{\nu_2} n_2(x))}{4|\xi'|_{\Gamma}^2}, \quad x \in \Gamma, \quad \xi' \in \mathbf{R}^{d-1}.$$

(3) *When λ is a pole of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$, the regular part of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ is singular integro-differential operator on $B(\lambda_0)^c$ with order -1 for (A-2-1) or -2 for (A-2-2). Its principal symbol is given by (2.42) or (2.43), respectively.*

(4) *For both of (A-2-1) or (A-2-2), $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ is Fredholm for $\lambda \in \mathbf{C} \setminus \{0\}$.*

Proof. Let n_1 and n_2 satisfy (A-2-1). In y -coordinates, we have $\tilde{A}_{1,0} = \tilde{A}_{2,0}$, $\tilde{A}_{1,1} = \tilde{A}_{2,1}$ and $\tilde{A}_{1,2} - \tilde{A}_{2,2} = -\lambda(n_1(y', 0) - n_2(y', 0))$. Then $E_{1,0} = E_{2,0} = e^{-|\xi'|_{\Gamma} y_d}$, $E_{1,1} = E_{2,1}$ and

$$(-\partial_d^2 + |\xi'|_{\Gamma}^2)(E_{1,2} - E_{2,2}) = \lambda(n_1(y', 0) - n_2(y', 0))e^{-|\xi'|_{\Gamma} y_d}.$$

A particular solution of this equation is

$$\frac{\lambda(n_1(y', 0) - n_2(y', 0))}{2|\xi'|_{\Gamma}} y_d e^{-|\xi'|_{\Gamma} y_d},$$

which vanishes at $y_d = 0$ and $y_d \rightarrow \infty$. Then we can take it as $E_{1,2} - E_{2,2}$, and $-\partial_d(E_{1,2} - E_{2,2})$ at $y_d = 0$ is the principal symbol of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$. In view of the assertion (1) in Corollary 2.9, we have the assertion (1).

Next we assume that n_1 and n_2 satisfy (A-2-2). As above, we have $\tilde{A}_{1,j} = \tilde{A}_{2,j}$ for $j = 0, 1, 2$, and $\tilde{A}_{1,3} - \tilde{A}_{2,3} = -\lambda(\partial_d n_1(y', 0) - \partial_d n_2(y', 0))y_d$. Then we have

$$E_{1,3} - E_{2,3} = \frac{\lambda}{4}(\partial_d n_1(y', 0) - \partial_d n_2(y', 0)) \frac{y_d}{|\xi'|_\Gamma} \left(y_d + \frac{1}{|\xi'|_\Gamma} \right) e^{-|\xi'|_\Gamma y_d}.$$

Hence we obtain the assertion (2).

In view of Corollary 2.3 and the assertion (2) in Corollary 2.9, we can show the assertion (3) by the similar way.

The ellipticity of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ implies that $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ is Fredholm for $\lambda \in \mathbf{C} \setminus \{0\}$. \square

3. INTERIOR TRANSMISSION EIGENVALUES

Here we prove main results. Let us list our assumptions again :

(A-1) M_1 and M_2 have a common boundary $\Gamma := \partial M_1 = \partial M_2$. Γ is a disjoint union of a finite number of connected and closed components. The metrics satisfy $g_1 = g_2$ on Γ .

(A-2) The metrics g_1, g_2 and the indices of refraction n_1, n_2 satisfy one of following two cases :

(A-2-1) For all $x \in \Gamma$, $\partial_{\nu_1}^m g_1^{ij}(x) = \partial_{\nu_2}^m g_2^{ij}(x)$ for $m \leq 2$, $i, j = 1, \dots, d$, and $n_1(x) \neq n_2(x)$,

or

(A-2-2) For all $x \in \Gamma$, $\partial_{\nu_1}^m g_1^{ij}(x) = \partial_{\nu_2}^m g_2^{ij}(x)$ for $m \leq 3$, $i, j = 1, \dots, d$, and $n_1(x) = n_2(x)$, $\partial_{\nu_1} n_1(x) \neq \partial_{\nu_2} n_2(x)$.

Throughout of §3, we suppose the above conditions.

3.1. Discreteness of the set of ITEs. For the proof of discreteness, we need to use the analytic Fredholm theory which was generalized by [2]. See also Appendix A in [16]. Let H_1 and H_2 are Hilbert spaces. We take a connected open domain $D \subset \mathbf{C}$. An operator valued function $A(z) : H_1 \rightarrow H_2$ for $z \in D \subset \mathbf{C}$ is finitely meromorphic if the principal part of the Laurent series at a pole of $A(z)$ is a finite rank operator. In particular, $\Lambda_k(\lambda) : H^{3/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$ is finitely meromorphic in $\mathbf{C} \setminus \{0\}$ as has been seen in Proposition 2.1.

Theorem 3.1. *Suppose an operator valued function $A(z) : H_1 \rightarrow H_2$, $z \in D$, is finitely meromorphic and Fredholm. If there exists its bounded inverse $A(z_0)^{-1} : H_2 \rightarrow H_1$ at a point $z_0 \in D$, then $A(z)^{-1}$ is finitely meromorphic and Fredholm in D .*

From the above theorem, if there exists a $\lambda \in \mathbf{C} \setminus \{0\}$ such that $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ is invertible, $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ is invertible in $\mathbf{C} \setminus (\{0\} \cup S')$ for a discrete subset S' of \mathbf{C} . Therefore, for the proof of the discreteness, we have only to show that $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ is invertible for some $\lambda \in \mathbf{C} \setminus \{0\}$.

We expand the symbol of A_k centered at $(z', 0) \in \partial\Omega^0$ by the same manner in §2.2. However, here we change the definition of homogeneous functions with generalized degree s by

$$(3.1) \quad F(t\kappa; t^{-1}y', t^{-1}y_d, t\xi', t\xi_d) = t^s F(\kappa; y', y_d, \xi', \xi_d), \quad t > 0, \quad \kappa = \sqrt{\lambda},$$

for $\lambda \in \mathbf{C} \setminus \{0\}$, taking a suitable branch of $\kappa = \sqrt{\lambda}$. We gather terms of the same generalized degree in the sense (3.1), and we denote the symbol in y -coordinates by

$$A_k(\kappa; y', y_d, \xi', \xi_d) = \sum_{m=0}^N \mathcal{A}_{k,m}(\kappa, z'; y' - z', y_d, \xi', \xi_d),$$

up to the remainder term where $\mathcal{A}_{k,m}$ is homogeneous of degree $2-m$. In particular, putting $\tilde{\mathcal{A}}_{k,m}^{(\lambda)} = \mathcal{A}_{k,m}(\kappa, z'; -i\partial_{\xi'}, y_d, \xi', i\partial_d)$, we have

$$(3.2) \quad \tilde{\mathcal{A}}_{k,0}^{(\lambda)} = -\partial_d^2 + |\xi'|_\Gamma^2 - \lambda n_k(z', 0),$$

$$(3.3) \quad \tilde{\mathcal{A}}_{k,1}^{(\lambda)} = \tilde{A}_{k,1} + \lambda \tilde{B}_{k,1}^{(\lambda)},$$

where $\tilde{A}_{k,1}$ is defined by (2.24) and

$$\tilde{B}_{k,1}^{(\lambda)} = i\nabla_{y'} n_k(z', 0) \cdot \nabla_{\xi'} - y_d \partial_d n_k(z', 0).$$

We denote by $\{E_{k,m}^{(\lambda)}\}_{m \geq 0}$ the solution of

$$(3.4) \quad \tilde{\mathcal{A}}_{k,0}^{(\lambda)} E_{k,0}^{(\lambda)} = 0,$$

$$(3.5) \quad \tilde{\mathcal{A}}_{k,0}^{(\lambda)} E_{k,m}^{(\lambda)} = - \sum_{n=0}^m \tilde{\mathcal{A}}_{k,n}^{(\lambda)} E_{k,m-n}^{(\lambda)}, \quad m \geq 1,$$

with the boundary condition $E_{k,0}^{(\lambda)}|_{y_d=0} = 1$, $E_{k,m}^{(\lambda)}|_{y_d=0} = 0$ for $m \geq 1$ and $E_{k,m}^{(\lambda)} \rightarrow 0$ as $y_d \rightarrow \infty$ for $m \geq 0$.

Let $P(\lambda)$ be a pseudo differential operator (or a singular integro-differential operator) which is parametrized by $\lambda \in \mathbf{C}$ such that the full symbol of $P(\lambda)$ is a asymptotic series of homogeneous functions in the sense (3.1). If the principal symbol does not vanish when $|\lambda| + |\xi'|_\Gamma^2 \neq 0$, we call $P(\lambda)$ *parameter elliptic*.

Lemma 3.2. *Let $\lambda \in \mathbf{C} \setminus \mathbf{R}_{\geq 0}$.*

(1) *We assume that (A-2-1) holds. Then $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ is parameter elliptic. Its principal symbol is*

$$(3.6) \quad \frac{-\lambda(n_1(x) - n_2(x))}{\sqrt{|\xi'|_\Gamma^2 - \lambda n_1(x)} + \sqrt{|\xi'|_\Gamma^2 - \lambda n_2(x)}}, \quad x \in \Gamma, \quad \xi' \in \mathbf{R}^{d-1}.$$

(2) *We assume that (A-2-2) holds. Then $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ is parameter elliptic. Its principal symbol is*

$$(3.7) \quad \frac{\lambda(\partial_{\nu_1} n_1(x) - \partial_{\nu_2} n_2(x))}{4(|\xi'|_\Gamma^2 - \lambda n(x))}, \quad x \in \Gamma, \quad \xi' \in \mathbf{R}^{d-1},$$

where $n(x) := n_1(x) = n_2(x)$ and $\tilde{g}^{ij}(x) := \tilde{g}_1^{ij}(x) = \tilde{g}_2^{ij}(x)$.

Proof. We fix an arbitrary point $(z', 0) \in \partial\Omega^0$. Suppose that (A-2-1) holds. Obviously we have

$$E_{k,0}^{(\lambda)}(z'; \xi', y_d) = \exp\left(-\sqrt{|\xi'|_\Gamma^2 - \lambda n_k(z', 0)} y_d\right).$$

Under the assumption, we also have $\tilde{\mathcal{A}}_{1,0}^{(\lambda)} \neq \tilde{\mathcal{A}}_{2,0}^{(\lambda)}$ so that $E_{1,0}^{(\lambda)} \neq E_{2,0}^{(\lambda)}$. Then the principal symbol $-\partial_d(E_{1,0}^{(\lambda)} - E_{2,0}^{(\lambda)})|_{y_d=0}$ of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ is given by (3.6).

Let us consider the case (A-2-2). In view of $n_1 = n_2 (= n)$ on Γ , we have $\tilde{\mathcal{A}}_{1,0}^{(\lambda)} = \tilde{\mathcal{A}}_{2,0}^{(\lambda)}$ so that

$$E_{1,0}^{(\lambda)}(z'; \xi', y_d) = E_{2,0}^{(\lambda)}(z'; \xi', y_d) = \exp \left(-\sqrt{|\xi'|_\Gamma^2 - \lambda n(z', 0)} y_d \right).$$

Since we have assumed (A-1) and (A-2-2), we have

$$\tilde{\mathcal{A}}_{1,1}^{(\lambda)} - \tilde{\mathcal{A}}_{2,1}^{(\lambda)} = -\lambda(\partial_d n_1(z', 0) - \partial_d n_2(z', 0)) y_d.$$

Then $E_{1,1}^{(\lambda)} - E_{2,1}^{(\lambda)}$ satisfies the equation

$$\begin{aligned} & (-\partial_d^2 + |\xi'|_\Gamma^2 - \lambda n(z', 0))(E_{1,1}^{(\lambda)} - E_{2,1}^{(\lambda)}) \\ & = \lambda(\partial_d n_1(z', 0) - \partial_d n_2(z', 0)) y_d \exp \left(-\sqrt{|\xi'|_\Gamma^2 - \lambda n(z', 0)} y_d \right). \end{aligned}$$

Precisely, we obtain

$$\begin{aligned} & E_{1,1}^{(\lambda)}(z'; \xi', y_d) - E_{2,1}^{(\lambda)}(z'; \xi', y_d) = -\frac{\lambda}{4}(\partial_d n_1(z', 0) - \partial_d n_2(z', 0)) \\ & \cdot \left(\frac{y_d^2}{\sqrt{|\xi'|_\Gamma^2 - \lambda n(z', 0)}} + \frac{y_d}{|\xi'|_\Gamma^2 - \lambda n(z', 0)} \right) \exp \left(-\sqrt{|\xi'|_\Gamma^2 - \lambda n(z', 0)} y_d \right). \end{aligned}$$

Then the principal symbol $-\partial_d(E_{1,1}^{(\lambda)} - E_{2,1}^{(\lambda)})|_{y_d=0}$ of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ is given by (3.7). \square

In view of Lemma 3.2, we can obtain a uniform estimate in λ of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$ and its inverse. In the following, we define the Hilbert space $H^{m,t}(\Gamma)$ for $t > 0$ by the norm

$$\|f\|_{H^{m,t}(\Gamma)}^2 = \|f\|_{H^m(\Gamma)}^2 + t^{2m} \|f\|_{L^2(\Gamma)}^2.$$

We take an arbitrary closed sector \mathbf{S}_0 centered at the origin such that $\mathbf{S}_0 \cap \mathbf{R}_{>0} = \emptyset$. We put $\mathbf{S}_0^e := \mathbf{S}_0 \cap \{\lambda \in \mathbf{C} ; |\lambda| \geq 1\}$. For the proof of the next result, see Theorem 4.4.6 in [1] and see also [6].

Theorem 3.3. *Let $\lambda \in \mathbf{S}_0^e$ and $|\lambda|$ be sufficiently large. We put $t = \sqrt{|\lambda|}$. Then for each $m \in \mathbf{R}$, the operator $\lambda^{-1}(\Lambda_1(\lambda) - \Lambda_2(\lambda)) : H^{m,t}(\Gamma) \rightarrow H^{m+s,t}(\Gamma)$ is invertible where $s = 1$ for (A-2-1) or $s = 2$ for (A-2-2).*

Now we have our first main theorem as a corollary of Theorems 3.1 and 3.3.

Theorem 3.4. *We assume (A-1) and one of (A-2-1) and (A-2-2). The set of ITEs consists of a discrete subset of \mathbf{C} with the only possible accumulation points at 0 and infinity. There exist at most finitely many ITEs in \mathbf{S}_0^e .*

3.2. Weyl type estimate for interior transmission eigenvalues. In the following, we use Weyl's asymptotic behavior for Dirichlet eigenvalues of $-n_k^{-1} \Delta_{g_k}$ on M_k . The following fact is a direct consequence of Theorem 1.2.1 in [14].

Theorem 3.5. *Let $\mathcal{O}_k(x) = \{\xi \in \mathbf{R}^d ; \sum_{i,j} g^{ij}(x) \xi_i \xi_j \leq n_k(x)\}$ for each $x \in M_k$ and*

$$v(\mathcal{O}_k(x)) := \int_{\mathcal{O}_k(x)} d\xi,$$

be the volume of $\mathcal{O}_k(x)$ associated by the Euclidean measure. Then $N_k(\lambda) := \#\{j ; \lambda_{k,j} \leq \lambda\}$ satisfies as $\lambda \rightarrow \infty$

$$(3.8) \quad N_k(\lambda) = V_k \lambda^{d/2} + O(\lambda^{(d-1)/2}), \quad V_k = (2\pi)^{-d} \int_{M_k} v(\mathcal{O}_k(x)) dV_k.$$

In view of (A-1), there exists a positive integer m such that

$$(3.9) \quad \Gamma = \bigcup_{l=1}^m \Gamma_l, \quad \Gamma_j \cap \Gamma_k = \emptyset \quad \text{if } j \neq k,$$

where each Γ_l is a closed and connected component of Γ . Taking an arbitrary point $x^{(0)} \in \Gamma_l$, we take a small neighborhood $V \subset \Gamma_l$ of $x^{(0)}$ and a sufficiently small open domain Ω which is diffeomorphic to $U_1 \cong U_2$ such that $\overline{U_1} \cap \Gamma_l = \overline{U_2} \cap \Gamma_l = V$ as has been defend in the beginning of §2.2. Then, identifying $x \in V$ with the corresponding point on $\partial\Omega^0$, the following function

$$(3.10) \quad \gamma_l(x) := \begin{cases} \operatorname{sgn}(n_2(y) - n_1(y)) & \text{for (A-2-1)-case} \\ -\operatorname{sgn}(\partial_{\nu_2} n_2(y) - \partial_{\nu_1} n_1(y)) & \text{for (A-2-2)-case} \end{cases}$$

for $y \in \Omega$ is well-defined constant function $\gamma_l(x) = +1$ or -1 in V . Obviously, the function $\gamma_l(x) = 1$ or -1 can be naturally extended on Γ_l . We define the function $\gamma(x)$ on Γ by

$$(3.11) \quad \gamma(x) = \gamma_l(x), \quad \text{if } x \in \Gamma_l.$$

In the following, we use an auxiliary operator defined by

$$(3.12) \quad B(\lambda) = \gamma D_\Gamma^{(1+s)/4} (\Lambda_1(\lambda) - \Lambda_2(\lambda)) D_\Gamma^{(1+s)/4}.$$

where $s = 1$ for (A-2-1) or $s = 2$ for (A-2-2). Here D_Γ is given by $D_\Gamma = -\Delta_\Gamma + 1$ where Δ_Γ is the Laplace-Beltrami operator on Γ . Then $B(\lambda)$ is a first order singular integro-differential operator when λ is not a pole of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$.

Lemma 3.6. (1) Suppose $\lambda \notin \{\lambda_{1,j}\} \cap \{\lambda_{2,j}\}$. Then $\lambda \in \mathbf{C}$ is an ITE if and only if $\operatorname{Ker} B(\lambda) \neq \{0\}$. The multiplicity of λ coincides with $\dim \operatorname{Ker} B(\lambda)$.

(2) Suppose $\lambda \in \{\lambda_{1,j}\} \cap \{\lambda_{2,j}\}$. Then $\lambda \in \mathbf{C}$ is an ITE if and only if $\operatorname{Ker} B(\lambda) \neq \{0\}$ or the range of $\gamma D_\Gamma^{(1+s)/4} Q_{1,\mathcal{L}(\lambda)} D_\Gamma^{(1+s)/4}$ and $\gamma D_\Gamma^{(1+s)/4} Q_{2,\mathcal{L}(\lambda)} D_\Gamma^{(1+s)/4}$ have a non trivial intersection. The multiplicity of λ coincides with the sum of $\dim \operatorname{Ker} B(\lambda)$ and the dimension of the above intersection.

Proof. Since $-\Delta_\Gamma + 1$ is invertible, the lemma is a direct consequence of Lemma 2.4. \square

Lemma 3.7. Let $\lambda \in \mathbf{C} \setminus \{0\}$ be not a pole of $B(\lambda)$.

(1) Then $B(\lambda)$ is a first order, symmetric and elliptic singular integro-differential operator. Its principal symbol is

$$(3.13) \quad -\frac{\lambda \gamma(n_1(x) - n_2(x))}{2} |\xi'|_\Gamma, \quad x \in \Gamma, \quad \xi' \in \mathbf{R}^{d-1},$$

for (A-2-1), or

$$(3.14) \quad \frac{\lambda \gamma(\partial_{\nu_1} n_1(x) - \partial_{\nu_2} n_2(x))}{4} |\xi'|_\Gamma, \quad x \in \Gamma, \quad \xi' \in \mathbf{R}^{d-1},$$

for (A-2-2), where $\tilde{g}^{ij}(x) = g_1^{ij}(x) = g_2^{ij}(x)$.

(2) For $\lambda \in \mathbf{R}_{>0}$, the spectrum of $B(\lambda)$ is discrete and consists of the set of real eigenvalues $\{\mu_j(\lambda)\}$.

Proof. We have the first assertion by direct computation using Lemma 2.10. From the first assertion, we also see the second assertion. \square

Since $B(\lambda)$ has a positive principal symbol and $B(\lambda)$ is meromorphic with respect to λ , we also have the following lemma. The proof is completely same of Lemma 2.4 and 2.5 in [11]. Note that, in view of (2.6), we define the residue $\text{res}_{\lambda=\lambda_0}\mu_j(\lambda)$ of $\mu_j(\lambda)$ at a pole λ_0 by the expansion

$$(3.15) \quad \mu_j(\lambda) = \frac{\text{res}_{\lambda=\lambda_0}\mu_j(\lambda)}{\lambda_0 - \lambda} + \tilde{\mu}_j(\lambda),$$

where $\tilde{\mu}_j(\lambda)$ is analytic in a small neighborhood of λ_0 .

Lemma 3.8. (1) For each compact interval $I \subset \mathbf{R}_{>0}$ such that any pole of $B(\lambda)$ are not included in I , there exists a constant $C(I) > 0$ such that $\mu_j(\lambda) \geq -C(I)$ for $\lambda \in I$, $j = 1, 2, \dots$.

(2) If $B(\lambda)$ is analytic in a neighborhood of λ_0 , all eigenvalues $\mu_j(\lambda)$ are analytic in this neighborhood. If λ_0 is a pole of $B(\lambda)$ and p is the rank of the residue of $B(\lambda)$ at λ_0 , p eigenvalues $\mu_j(\lambda)$ and its eigenfunctions have a pole at λ_0 . Moreover, $\text{res}_{\lambda=\lambda_0}\mu_j(\lambda)$ are eigenvalues of $\text{res}_{\lambda=\lambda_0}B(\lambda)$.

We choose a small constant $\alpha \in (0, \min\{\lambda_{1,1}, \lambda_{2,1}\})$. We define counting function with multiplicities taken into account :

$$(3.16) \quad N_T(\lambda) = \#\{j ; \alpha < \lambda_j^T \leq \lambda\},$$

where $\lambda_1^T \leq \lambda_2^T \leq \dots$ are ITEs included in (α, ∞) .

Now we consider the relation between $\{\lambda_j^T\}$ and $\{\mu_j(\lambda)\}$ for $\lambda \in (\alpha, \infty)$. Roughly speaking, we can evaluate $N_T(\lambda)$ by the number of the singular spectral points and the number of λ satisfying $\mu_j(\lambda) = 0$. We put

$$(3.17) \quad N_-(\lambda) = \#\{j ; \mu_j(\lambda) < 0\}, \quad \lambda \notin \{\lambda_j^T\} \cup \{\lambda_{1,j}\} \cup \{\lambda_{2,j}\},$$

Assume that λ' moves from α to ∞ . Since $\mu_j(\lambda')$ is meromorphic with respect to λ' , $N_-(\lambda')$ changes only when some $\mu_j(\lambda')$ pass through 0 or λ' passes through a pole of $B(\lambda')$. When λ' moves from α to $\lambda > \alpha$, we denote by $\mathcal{N}_0(\lambda)$ the change of $N_-(\lambda) - N_-(\alpha)$ due to the first case, and $\mathcal{N}_{-\infty}(\lambda)$ as the change due to the second case i.e.

$$(3.18) \quad N_-(\lambda) - N_-(\alpha) = \mathcal{N}_0(\lambda) + \mathcal{N}_{-\infty}(\lambda).$$

For a pole λ_0 of $B(\lambda)$, we put

$$(3.19) \quad \delta\mathcal{N}_{-\infty}(\lambda_0) = N_-(\lambda_0 + \epsilon) - N_-(\lambda_0 - \epsilon),$$

for any $\epsilon > 0$.

Lemma 3.9. Let $\lambda_0 \in \mathbf{R}_{>0}$ be a pole of $B(\lambda)$. We have $\delta\mathcal{N}_{-\infty}(\lambda_0) = s_+(\lambda_0) - s_-(\lambda_0)$ for $s_{\pm}(\lambda_0) = \#\{j ; \pm\text{res}_{\lambda=\lambda_0}\mu_j(\lambda) > 0\}$.

Proof. In view of Lemma 3.8, some eigenvalues $\mu_j(\lambda)$ have a pole at λ_0 . If $\pm\text{res}_{\lambda=\lambda_0}\mu_j(\lambda) > 0$, we have $\pm\mu_j(\lambda) \rightarrow \mp\infty$ as $\lambda \rightarrow \lambda_0 + 0$ and $\pm\mu_j(\lambda) \rightarrow \pm\infty$ as $\lambda \rightarrow \lambda_0 - 0$, respectively. Then the number of negative eigenvalues decreases for $\text{res}_{\lambda=\lambda_0}\mu_j(\lambda) < 0$ and increases for $\text{res}_{\lambda=\lambda_0}\mu_j(\lambda) > 0$ when λ passes through λ_0 from α . This implies the lemma. \square

Lemma 3.10. *If $\lambda_0 \in \mathbf{R}_{>0}$ is a pole of $\Lambda_k(\lambda)$, the residue $Q_{k,\mathcal{L}(\lambda_0)}$ is negative.*

Proof. Recall that $B_k(\lambda_0)$ is the subspace of $L^2(\Gamma)$ spanned by $\partial_{\nu_k} \phi_{k,j}$ for $j \in \mathcal{L}(\lambda_0)$. In view of (2.7), we have for $0 \neq f \in B_k(\lambda_0)$

$$(Q_{k,\mathcal{L}(\lambda_0)} f, f)_{L^2(\Gamma)} = - \sum_{j \in \mathcal{L}(\lambda_0)} |(\partial_{\nu_k} \phi_{k,j}, f)_{L^2(\Gamma)}|^2 < 0.$$

Then we have $Q_{k,\mathcal{L}(\lambda_0)} < 0$. \square

We denote by Γ_{\pm} a union of connected components Γ_l such that $\gamma|_{\Gamma_l} = \pm 1$, respectively. Let $\lambda_0 \in \{\lambda_{k,j}\}$. We put $m_k(\lambda_0) = \dim \text{Ran} Q_{k,\mathcal{L}(\lambda_0)}$ and $m(\lambda_0) = \dim(\text{Ran} Q_{1,\mathcal{L}(\lambda_0)} \cap \text{Ran} Q_{2,\mathcal{L}(\lambda_0)})$, and we split $m_k(\lambda_0) = m_k^+(\lambda_0) + m_k^-(\lambda_0)$ and $m(\lambda_0) = m^+(\lambda_0) + m^-(\lambda_0)$ with $m_k^{\pm}(\lambda_0) = \dim(\text{Ran} Q_{k,\mathcal{L}(\lambda_0)} \cap L^2(\Gamma_{\pm}))$ and $m^{\pm}(\lambda_0) = \dim(\text{Ran} Q_{1,\mathcal{L}(\lambda_0)} \cap \text{Ran} Q_{2,\mathcal{L}(\lambda_0)} \cap L^2(\Gamma_{\pm}))$ for a pole $\lambda_0 \in \mathbf{R}_{>0}$ of $B(\lambda)$.

Lemma 3.11. *Let $\lambda_0 \in \mathbf{R}_{>0}$ be a pole of $B(\lambda)$.*

(1) *If $\lambda_0 \notin \{\lambda_{1,j}\} \cap \{\lambda_{2,j}\}$, we have $\delta \mathcal{N}_{-\infty}(\lambda_0) = (m_2^+(\lambda_0) - m_2^-(\lambda_0)) - (m_1^+(\lambda_0) - m_1^-(\lambda_0))$.*

(2) *If $\lambda_0 \in \{\lambda_{1,j}\} \cap \{\lambda_{2,j}\}$, we have $|\delta \mathcal{N}_{-\infty}(\lambda_0) - ((m_2^+(\lambda_0) - m_2^-(\lambda_0)) - (m_1^+(\lambda_0) - m_1^-(\lambda_0)))| \leq m(\lambda_0)$.*

Proof. First we prove the assertion (1). Suppose $\lambda_0 \in \{\lambda_{1,j}\}$. We can expand $B(\lambda)$ in a small neighborhood of λ_0 as

$$B(\lambda) = \frac{\gamma D_{\Gamma}^{(1+s)/4} Q_{1,\mathcal{L}(\lambda_0)} D_{\Gamma}^{(1+s)/4}}{\lambda_0 - \lambda} + \tilde{H}_{\lambda_0}(\lambda),$$

where $\tilde{H}_{\lambda_0}(\lambda)$ is analytic. From Lemma 3.10, we have $Q_{1,\mathcal{L}(\lambda_0)} < 0$ and also $D_{\Gamma}^{(1+s)/4} Q_{1,\mathcal{L}(\lambda_0)} D_{\Gamma}^{(1+s)/4} < 0$ so that $D_{\Gamma}^{(1+s)/4} Q_{1,\mathcal{L}(\lambda_0)} D_{\Gamma}^{(1+s)/4}$ has exactly $m_1(\lambda_0)$ strictly negative eigenvalues. Then, in view of the assertion (2) in Lemma 3.8 and Lemma 3.9, we have $s_{\pm}(\lambda_0) = m_1^{\mp}(\lambda_0)$, so $\delta \mathcal{N}_{-\infty}(\lambda_0) = m_1^-(\lambda_0) - m_1^+(\lambda_0)$ with $m_2(\lambda_0) = 0$. For the case $\lambda_0 \in \{\lambda_{2,j}\}$, we can see $\delta \mathcal{N}_{-\infty}(\lambda_0) = m_2^+(\lambda_0) - m_2^-(\lambda_0)$ with $m_1(\lambda_0) = 0$ by the similar way. Plugging these two cases, we obtain the assertion (1).

Let us prove the assertion (2). Suppose $\lambda_0 = \lambda_{1,i_1} = \lambda_{2,i_2}$ for $\lambda_{1,i_1} \in \{\lambda_{1,j}\}$ and $\lambda_{2,i_2} \in \{\lambda_{2,j}\}$. Then we have the following representation in a small neighborhood of λ_0

$$B(\lambda) = \frac{\gamma Q_{\lambda_0}}{\lambda_0 - \lambda} + \tilde{H}_{\lambda_0}(\lambda),$$

with $Q_{\lambda_0} = D_{\Gamma}^{(1+s)/4} (Q_{1,\mathcal{L}(\lambda_{1,i_1})} - Q_{2,\mathcal{L}(\lambda_{2,i_2})}) D_{\Gamma}^{(1+s)/4}$. We see that $Q_{\lambda_0} < 0$ on $B_1(\lambda_{1,i_1}) \cap B_2(\lambda_{2,i_2})^{\perp}$ and $Q_{\lambda_0} > 0$ on $B_1(\lambda_{1,i_1})^{\perp} \cap B_2(\lambda_{2,i_2})$. Then we have $m_1^-(\lambda_0) + m_2^+(\lambda_0) - m(\lambda_0) \leq s_+(\lambda_0) \leq m_1^-(\lambda_0) + m_2^+(\lambda_0)$ and $m_1^+(\lambda_0) + m_2^-(\lambda_0) - m(\lambda_0) \leq s_-(\lambda_0) \leq m_1^+(\lambda_0) + m_2^-(\lambda_0)$. These inequalities and Lemma 3.9 imply the assertion (2). \square

Now we have arrived at our main result on the Weyl type lower bound for $N_T(\lambda)$.

Theorem 3.12. *We assume (A-1) and one of (A-2-1) and (A-2-2). For large $\lambda \in \mathbf{R}_{>0}$, we have*

$$(3.20) \quad N_T(\lambda) \geq \sum_{\alpha < \lambda' \leq \lambda} ((m_1^+(\lambda') - m_1^-(\lambda')) - (m_2^+(\lambda') - m_2^-(\lambda'))) - N_-(\alpha),$$

where the summation is taken over poles $\lambda' \in (\alpha, \lambda]$ of $\Lambda_1(\lambda) - \Lambda_2(\lambda)$. Moreover, if $\Gamma_+ = \emptyset$ or $\Gamma_- = \emptyset$ and $V_1 - V_2 > 0$ or $-(V_1 - V_2) > 0$ respectively where $V_1, V_2 > 0$ are defined in (3.8), $N_T(\lambda)$ satisfies asymptotically as $\lambda \rightarrow \infty$

$$(3.21) \quad N_T(\lambda) \geq \mp(V_1 - V_2)\lambda^{d/2} + O(\lambda^{(d-1)/2}).$$

Proof. We prove for the case $\{\lambda_{1,j}\} \cap \{\lambda_{2,j}\} \neq \emptyset$. For $\{\lambda_{1,j}\} \cap \{\lambda_{2,j}\} = \emptyset$, the proof is similar and can be slightly simplified. Letting us recall that we call λ is a singular spectral point when λ satisfies the latter condition of the assertion (2) of Lemma 2.4, we put

$$N_{sng}(\lambda) = \#\{\text{singular spectral points} \in (\alpha, \lambda] \subset \mathbf{R}_{>0}\}.$$

Here $N_{sng}(\lambda)$ counts the number of singular spectral points with multiplicities taken into account. Note that $\mathcal{N}_0(\lambda) + N_{sng}(\lambda) \leq N_T(\lambda)$ by the definition of $\mathcal{N}_0(\lambda)$ and Lemma 3.6. We take the summation of $|\delta\mathcal{N}_\infty(\lambda') - ((m_2^+(\lambda') - m_2^-(\lambda')) - (m_1^+(\lambda') - m_1^-(\lambda')))| \leq m(\lambda')$ over all poles λ' of $B(\lambda)$ in $(\alpha, \lambda]$. Then we have

$$\left| \mathcal{N}_\infty(\lambda) - \sum_{\alpha < \lambda' \leq \lambda} ((m_2^+(\lambda') - m_2^-(\lambda')) - (m_1^+(\lambda') - m_1^-(\lambda'))) \right| \leq N_{sng}(\lambda).$$

See also Remark of Proposition 2.1. Plugging this inequality and (3.18), we have

$$\begin{aligned} N_-(\lambda) - N_-(\alpha) + \sum_{\alpha < \lambda' \leq \lambda} ((m_1^+(\lambda') - m_1^-(\lambda')) - (m_2^+(\lambda') - m_2^-(\lambda'))) \\ \leq \mathcal{N}_0(\lambda) + N_{sng}(\lambda) \leq N_T(\lambda). \end{aligned}$$

Since $N_-(\lambda) \geq 0$, we obtain (3.20).

If $\Gamma_\pm = \emptyset$, we see $m_1(\lambda') = m_1^\mp(\lambda')$ and $m_2(\lambda') = m_2^\mp(\lambda')$ for any poles λ' , respectively, so that we have from (3.20)

$$N_T(\lambda) \geq \mp(N_1(\lambda) - N_2(\lambda)) - N_-(\alpha).$$

The asymptotic estimate (3.21) is a direct consequence of (3.20) and Theorem 3.5. \square

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